Math 564: Real analysis and measure theory
Lecture 8

Terminology. Let $(X, \mathcal{B}, \mu)$ be a measure space and P be a property of points in X (e.g. being transcendental for $X := \mathbb{R}$). Then we say that

P holds a.e. (almost everywhere) in X
or a.e. (almost every) $x \in X$ satisfies P
or P holds a.s. (almost surely)

$\Bigg\}$ if $\{x \in X : x \text{ satisfies } P\}$ is conull.

Measure exhaustion.

In a measure space, call a collection $\mathcal{C}$ of sets **almost disjoint** if the pairwise intersections of sets in $\mathcal{C}$ are null.

Ctbl pigeonhole principle (for σ-finite measures). Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Then any almost disjoint collection $\mathcal{C}$ of μ-measurable positive measure sets is ctbl.

Proof. We first prove this assuming $\mu(X) < \infty$. Then for each $n \in \mathbb{N}^+$, the set

$$\mathcal{C}_n := \{C \in \mathcal{C} : \mu(C) \geq \tfrac{1}{n}\}$$

is finite (in fact, $\leq n \cdot \mu(X)$ elements) and $\mathcal{C} = \bigcup_{n \in \mathbb{N}^+} \mathcal{C}_n$, so $\mathcal{C}$ is ctbl.

For the general σ-finite case, let $X = \bigsqcup_{n \in \mathbb{N}} X_n$ where each $X_n \in \mathcal{B}$ is of finite measure. And define

$$\mathcal{D}_n := \{C \in \mathcal{C} : \mu(C \cap X_n) > 0\}.$$

Then by the finite case, each $\mathcal{D}_n$ is ctbl and $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$, so $\mathcal{C}$ is ctbl. $\square$

Transfinite measure exhaustion. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let $(A_\alpha)_{\alpha < \omega_1}$ be an increasing sequence of μ-measurable sets, where ω_1 is the first unctbl ordinal. Then the sequence almost stabilizes at some ctbl ordinal γ, i.e. $\forall \alpha \geq \gamma$, $A_\alpha =_\mu A_\gamma$.

Proof. We disjointify: $A'_\alpha := A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$, so $\{A'_\alpha\}_{\alpha<\omega_1}$ is an almost disjoint collection, hence all but ctbly many of A'_α are null by ctbl pigeonhole, i.e. $\exists$ ctbl ordinal γ such that A'_α is null for all $\alpha > \gamma$, hence $A_\alpha =_\mu A_\gamma$ because $A_\alpha \setminus A_\gamma = \bigcup_{\gamma<\beta<\alpha} A'_\beta$ is null being a ctbl union of null sets. $\square$

Remark. This allows to run transfinite algorithms which at each step handle a positive measure set. Then we know the algorithm will stop at a ctbl stage, having handled a conull set.

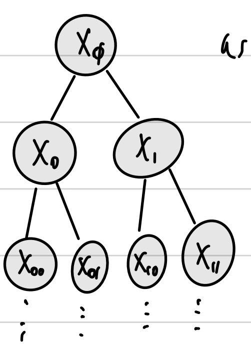
We now discuss an important application. In a measure space with atoms, we can't achieve every value of measure between 0 and $\mu(X)$, but this is the only obstruction.

Sierpinski's theorem. In an atomless measure space $(X, \mathcal{B}, \mu)$, every value $0 < r \leq \mu(X)$ is achieved, i.e. there is $B \in \mathcal{B}$ with $\mu(B) = r$.

Proof. First let's prove a more humble statement:

Claim 1. Every positive measure set Y contains positive measure sets of arbitrarily small measure.

Pf of Claim. Y is not an atom so there must be $X_\emptyset \subseteq Y$ with $\mu(X_\emptyset) < \mu(Y)$. We build a sequence $(X_s)_{s \in 2^{<\mathbb{N}}}$ of positive measure sets such that $X_s = X_{s0} \sqcup X_{s1}$ as follows: if X_s is defined, it's not an atom, so there is $X_{s0} \subseteq X_s$ in $\mathcal{B}$ with $0 < \mu(X_{s0}) < \mu(X_s)$. Let $X_{s1} := X_s \setminus X_{s0}$.



For each $s \in 2^{<\mathbb{N}}$, one of X_{s0} and X_{s1} has measure $\leq \frac{1}{2}\mu(X_s)$, which gives an infinite branch $(X_{s_n})_{n\in\mathbb{N}}$ in the tree of positive measure sets with $\mu(X_{s_n}) \leq \frac{1}{2^n} \cdot \mu(X_\emptyset)$. $\square$ (Claim 1)

Iteratively using Claim 1, we now explicitly build a set $B \in \mathcal{B}$ with $\mu(B) = r$.

Proof via transfinite exhaustion. Define a sequence $(A_\alpha)_{\alpha<\omega_1} \subseteq \mathcal{B}$ of pairwise disjoint sets such that $\mu(\bigsqcup_{\alpha<\beta} A_\alpha) \leq r$ for each $\beta<\omega_1$, by induction as follows: if $(A_\alpha)_{\alpha<\beta}$ is already defined, let A_β be a positive measure subset of $X \setminus \bigcup_{\alpha<\beta} A_\alpha$ of measure $\leq r - \mu(\bigsqcup_{\alpha<\beta} A_\alpha)$ if $r-\mu(\bigsqcup_{\alpha<\beta} A_\alpha)>0$; otherwise put $A_\beta := \emptyset$. Now the proof of ctbl pigeonhole for measures (using the condition $\mu(\bigsqcup_{\alpha<\beta} A_\alpha) \leq r$ $\forall \beta<\omega_1$ instead of the finiteness of μ) gives that all but ctbly many of the A_α are null, i.e. $\exists \beta<\omega_1$ with A_α null for all $\alpha \geq \beta$. Thus, $\mu(\bigsqcup_{\alpha<\beta} A_\alpha) = r$. $\square$

Proof via $\frac{1}{2}$-greedy algorithm. We inductively build a sequence $(B_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$ of pairwise disjoint sets such that $\mu(\bigsqcup_{i\leq n} B_i) \leq r$ as follows: suppose $(B_i)_{i<n}$ is defined and take $B_n \in \mathcal{B}$ to be any set with

$$\mu(B_n) \geq \frac{1}{2} \sup\left\{B \in \mathcal{B} : B \subseteq X \setminus \bigsqcup_{i<n} B_i \text{ and } \mu(B) \leq r - \mu\left(\bigsqcup_{i<n} B_i\right)\right\}.$$

Now that $(B_n)_{n\in\mathbb{N}}$ is defined, monotone convergence implies $\sum_{n\in\mathbb{N}} \mu(B_n) = \mu\left(\bigsqcup_{n\in\mathbb{N}} B_n\right) \leq r$, in particular, $\lim_{n\in\mathbb{N}} \mu(B_n) = 0$. We now check that the set

$$B_\infty := \bigsqcup_{n\in\mathbb{N}} B_n$$

has measure $= r$. Indeed, otherwise, $\mu(B_\infty) < r$, so by Claim 1, there is $B' \subseteq X \setminus B_\infty$ in $\mathcal{B}$ such that $0 < \mu(B') \leq r - \mu(B_\infty)$. But taking a large enough $n \in \mathbb{N}$ so that $\mu(B_n) < \frac{1}{2}\mu(B')$, we get a contradiction with the choice of B_n. $\square$

Approximating measurable sets

99% lemma. We begin with a basic observation.

Observation (percentage of carrots in soup). Let $(X, \mathcal{B}, \mu)$ be a measure space and let A, B be μ-measurable sets with $0 < \mu(B) < \infty$. Then for any (percentage) $p \in [0,1]$ and any (finite or ctbl) partition $B = \bigsqcup_{n<N} B_n$, where $N \in \mathbb{N} \cup \{\infty\}$, we have

$$\frac{\mu(A\cap B)}{\mu(B)} \geq p \implies \frac{\mu(A\cap B_n)}{\mu(B_n)} \geq p \text{ for some } n \in \mathbb{N}.$$

Proof. $\dfrac{\mu(A\cap B)}{\mu(B)} = \sum \dfrac{\mu(B_n)}{\mu(B)} \cdot \dfrac{\mu(A\cap B_n)}{\mu(B_n)}$, where $\sum_{n\in\mathbb{N}} \dfrac{\mu(B_n)}{\mu(B)} = 1$, so it's a convex combination. $\square$